ISI B. Math.
Physics I
Mid Semetral Exam
Total Marks: 100

Answer any five questions. All questions carry equal marks.

1. Consider the motion of a particle of mass m under the influence of a force $\mathbf{F} = -k\mathbf{r}$ where k is a positive constant and $\mathbf{r}$ is the position vector of the particle.
(a) Prove that the motion of the particle lies in a plane. (4)
(b) Find the position of the particle as a function of time, assuming that at $t = 0, x = a, y = 0$ and $v_x = 0, v_y = v_0$. (6)
(c) Show that the orbit is an ellipse.(4)
(d) Find the period.(3)
(e) Does the motion of the particle obey Kepler's laws of planetary motion?(3)

2. Consider a pendulum of length l and a bob of mass m at its end moving through oil. The massive bob undergoes small oscillations, but the oil retards the bob's motion with a resistive force proportional to the speed with $F_{\text{res}} = 2m\sqrt{\frac{g}{l}}(l\dot{\theta})$. The bob iis initially pulled back at $t = 0$ with $\theta = \alpha$ and $\dot{\theta} = 0$.
(a) Find the angular displacement θ and the velocity $\dot{\theta}$ as a function of time. (8)
(b) Show that at late times such that $\sqrt{\frac{g}{l}}t \gg 1$ the total mechanical energy of the pendulum decreases as (const)$\times t^2 e^{-2\sqrt{\frac{g}{l}}t}$ (6)
(c) Now assume that the pendulum is taken out of oil and put in a friction free medium amd started out with the same initial conditions. Make a sketch of the plot of θ vs $\dot{\theta}$ for a given energy fixed by the initial conditions. Such a plot is known as a phase space plot.(6)

3. (a) Check whether the following forces are conservative. If the answer is yes, find the potential energy for them. (8)
(i) $\mathbf{F} = \frac{\alpha(t)\hat{\mathbf{r}}}{r^2}$ where $\alpha(t)$ is a given function of time.
(ii) $\mathbf{F} = (6abz^3y - 20bx^3y^2)\mathbf{i} + (6abxz^3 - 10bx^4y)\mathbf{j} + 18abxz^2y\mathbf{k}$
where a, b are constants.

(b)The potential energy of a particle is given by

$$V(x, y, z) = \frac{k}{\sqrt{x^2 + y^2 + z^2}}$$

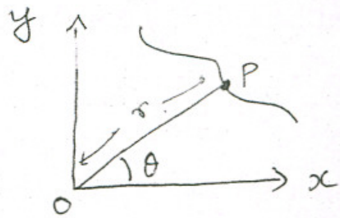
where k is a constant. Find the force on the particle. Is the force central? (4)

(c) A particle of mass $\sqrt{2}m$ collides elastically with a target particle of mass m at rest.

(i) Write down the appropriate equations of conservation of momentum and energy for such a collision and find an expression for the ratio of the final momentum to the initial momentum of the particle of mass $\sqrt{2}m$ in terms of m and θ, the angle of scattering of the particle. (4)

(ii) Using (i) show that the maximum angle through which the incident particle can be scattered is 45. (4)

4. (a) Consider a particle moving in a plane as shown.



In polar coordinates r and θ are some functions of time, and the position vector is given by $\vec{r} = r\hat{r}$. Show that (6)

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

and

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

(b) Suppose the above particle of mass m is moving under the influence of a central force. Which are the physical quantities that will be conserved during the motion ? (2)

(c) Write down the equations of motion of the particle moving under the influence of a central force $F(r)$ using the results from (a). (4)

(d) If $u = \frac{1}{r}$, Show that the equation of motion can be written as

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F(1/u)$$

where L is the angular momentum of the system (4).
(e) If the trajectory of the particle is given by $r\theta = constant$, find the form of the force as a function of r (4).

5. Suppose that the sun were surrounded by a dust cloud of uniform density ρ which extended at least as far as the orbital radius of the earth. The effect of the dust cloud is to modify the gravitational force experienced by the earth, so that the potential energy of the earth (neglecting the effect of the other planets)

$$U(r) = -\frac{GMm}{r} + \frac{1}{2}kr^2$$

where M is the mass of the sun, m is the mass of the earth, G is the gravitational constant, and $k = 4\pi\rho mG/3$.
(a) From the potential, find the force $\mathbf{F}$ acting upon the earth. (4)
(b) Make a careful sketch of the effective potential $U_{eff}(r)$. On your sketch, indicate (i)energy E_0 and radius r_0 of a circular orbit, and (ii) the energy E_1 and turning points r_1 and r_2 of an orbit which is not circular. (6)

(c) Assume that the earth is in a circular orbit of radius r_0 about the sun. Derive the equation satisfied by r_0 in terms of the angular momentum l and the constants m, M, G and k. You need not solve the equation.(3)
(d) Find the frequency of small oscillations ω_r about the circular orbit of radius r_0. You should find that your result can be written as

$$\omega_r = \sqrt{\omega_0^2 + \frac{3k}{m}}$$

where ω_0 is the angular velocity of revolution around the sun for a circular orbit. (4)
(e) Finally, by assuming k is small, show that the precession frequency ω_p for a nearly circular orbit $= \frac{3k}{2m\omega_0}$, recalling that the precession frequency is given by the difference between the small oscillation frequency and the orbital frequency of the earth. (3)

6. Consider the one dimensional potential

$$U(x) = -Wd^2\frac{(x^2 + d^2)}{(x^4 + 8d^4)}$$

(a) Where are the equilibrium points? (4)
(b) Are they stable or unstable? (4)
(c) Sketch the potential.(4)
(d) Is the motion bounded or unbounded ?(4)
(e) Find the turning points for $E = -\frac{W}{8}$. The value of W is a positive constant and so is d .(4)
(Hint: Define $Z(y) = \frac{U(x)}{W}$ where $y = \frac{x}{d}$ and sketch $Z(y)$ vs y to sketch the potential. Use parts (a) and (b) as guides to sketching the potential.)